1. Injectivity

1.1. **Functional Notation.** Let A and B be sets. The notation $f : A \to B$ means "f is a function whose domain is the set A and whose codomain is the set B." To shorten this, we say "f maps A into B".

In the following examples, let f have the natural domain and let the codomain equal the range.

- $f(x) = x^2$ $\quad f : \mathbb{R} \to [0, \infty)$
- $f(x) = x^3$ $\quad f : \mathbb{R} \to \mathbb{R}$
- $f(x) = \sqrt{x}$ $\quad f : [0, \infty) \to [0, \infty)$
- $f(x) = 2^x$ $\quad f : \mathbb{R} \to (0, \infty)$

2. One-to-one Functions

Definition 1. Let $f : A \to B$. We say that f is *one-to-one* (or *injective*) if, for every $a_1, a_2 \in A$, we have

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2.$$

That is, if a_1 and a_2 are points in the domain, and if a_1 is different from a_2, then $f(a_1)$ is different from a_2. In other words, a function is one-to-one if different points in the domain get mapped to different points in the range.

For example, the function $f(x) = x^2$ is NOT one-to-one, because (for example) 2 and -2 are two different points that get mapped to the same point, namely 4.

However, the function $f(x) = x^3$ IS one-to-one, because every real number has a unique cube root.

2.0.1. *Horizontal Line Test.* First, let us recall the *vertical line test*:

Proposition 1. (Vertical Line Test)
A subset of $\mathbb{R}^2$ is the graph of a function if and only if every vertical line intersects the graph at most once.

The *horizontal line test* tells when we have a one-to-one function.

Proposition 2. (Horizontal Line Test)
Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$. Then f is one-to-one if and only if every horizontal line intersects its graph at most once.

To see this, suppose that the horizontal line $y = c$ (for some real number c) intersects the graph of f at the distinct points (a, c) and (b, c). Then $f(a) = f(b) = c$, so the function is not one-to-one.

2.1. **Increasing and decreasing Functions.**

Definition 2. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$.

We say that f is *increasing on* D if, for every $x_1, x_2 \in D$, we have

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

We say that f is *decreasing on* D if, for every $x_1, x_2 \in D$, we have

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

A function is *monotone* if it is increasing or decreasing.

Proposition 3. *Monotone functions are one-to-one.*

2.2. **Exponential Functions.** Let a be a positive real numbers, and let $f(x) = a^x$.

- If $a > 1$, then f is an increasing function.
- If $0 < a < 1$, then f is a decreasing function.
- If $a = 1$, then f is a constant function.

Thus, as lone as $a \neq 1$, an exponential function is one-to-one. This means that

$$a^{x_1} = a^{x_2} \quad \Rightarrow \quad x_1 = x_2.$$

We can use this fact to solve equations.

Example 1. Solve $9^{x-4} = 27$.

Solution. First, we find a common base. Since $9 = 3^2$ and $27 = 3^3$, let $b = 3$ be the common base.

Thus $(3^2)^{x-4} = 9^{x-4} = 27 = 3^3$.

Using a property of exponents, this implies that $3^{2x-8} = 3^3$.

Since the function $f(x) = 3^x$ is injective, this implies that $2x - 8 = 3$.

Thus $2x = 11$, so $x = \dfrac{11}{2}$. □

2.3. **Exercises.**

Problem 1. Find all real numbers x which satisfy the following equations.

(a) $8^{2x-5} = 32^{x+7}$
(b) $125^{x^2-4} = 25^{-3x}$
(c) $343^{x^2-3x-8} = 49^{x^2-2x}$